UNIVERSITÄT
DES
SAARLANDES

ZBI ZENTRUM FÜR BIOINFORMATIK

Locality Sensitive Hashing

Algorithms for Sequence Analysis

Sven Rahmann
based on slides by Ali Ghaffaari

Summer 2021

Overview

Previous lectures

- Various **index structures for strings**:
  Suffix trees, suffix arrays, BWT, FM index, q-gram (k-mer) index
- Employ these indexes for exact and approximate search, **read mapping**

Today's lecture

- Finding exact and approximate matches by **hashing techniques**
- k-mers: encoding vs. hashing
- locality sensitive hashing (in general)
- min-hashing (on k-mers)

Part I: Remarks on the k-mer index

Implementations for a k-mer Index

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  Suffix array `pos` is useful for **every value of** k.

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- Today: Implementation as a hash table

Indexing *k*-mers: Hash Map

Example: Building a 3-mer index

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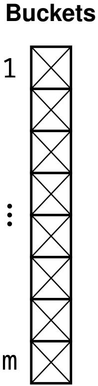
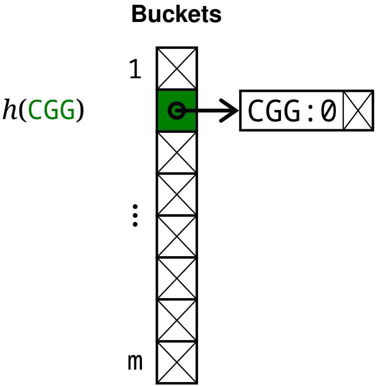


Indexing *k*-mers: Hash Map

Example: Building a 3-mer index

```
0123456789
```

$T =$ `CGGCATCATG`



Indexing k-mers: Hash Map

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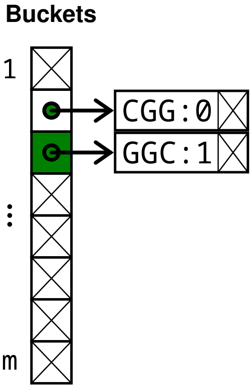
T = CGGCATCATG
```

$h(\texttt{GGC})$



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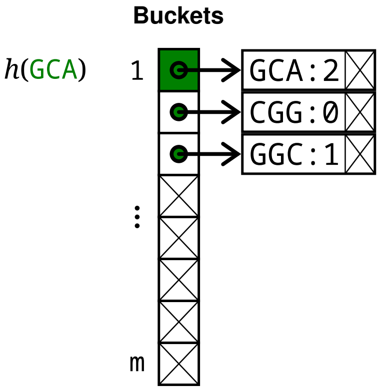

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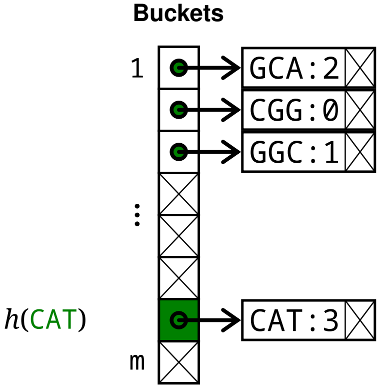

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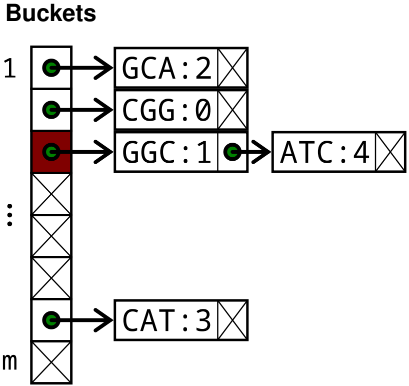

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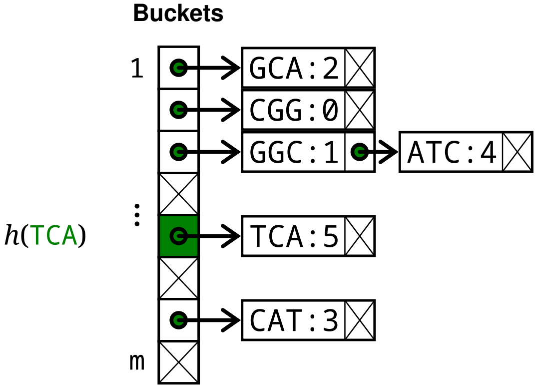

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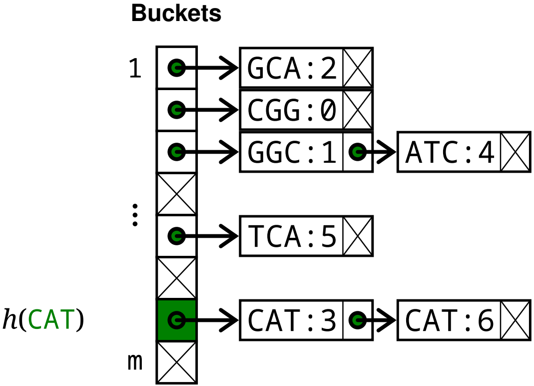

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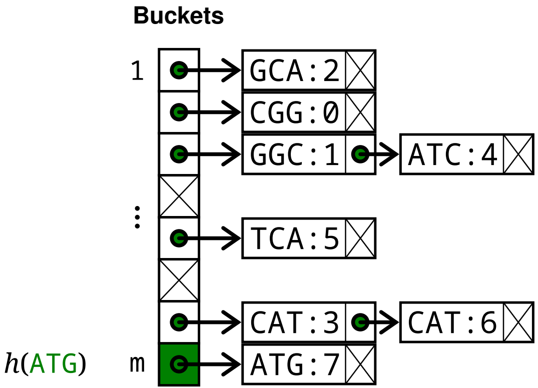

Querying a *k*-mer Index: Hash Map

Example: Querying an index

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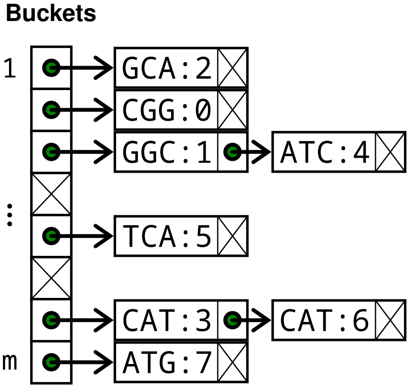

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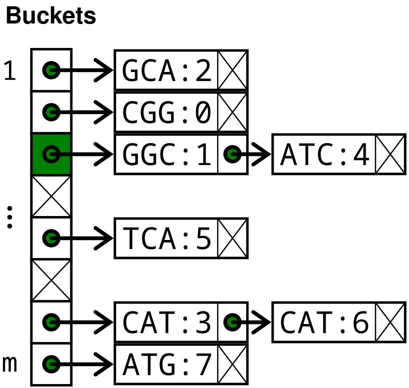

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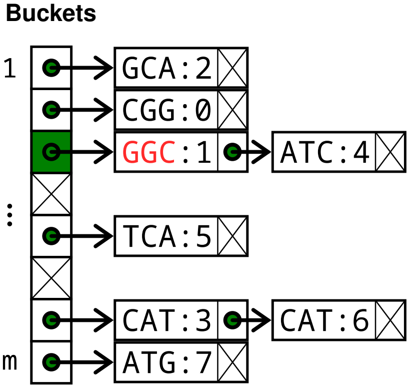

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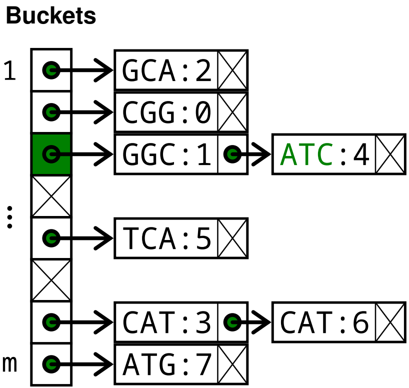

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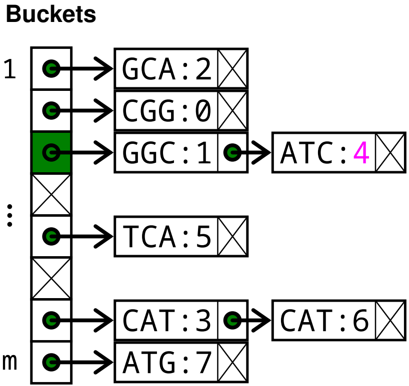

Encoding vs. Hashing

Encoding

- Assign a unique integer in $[0, 4^k[$ to each k-mer (e.g., base-4 encoding).
- Bijective map $\Sigma_{\mathrm{DNA}} \to \{0, \ldots, 4^k - 1\}$.
- Useful if $n = \Theta(4^k)$: Size of `pos`: n; size of `start`: 4^k.

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Hashing

- Any (non-bijective) function $\Sigma^k \to \{0, \ldots, p - 1\}$ for integer p (**address space**).
- Useful if $n \approx p \ll 4^k$ (large k).
- **Disadvantages:**
  - storage of k-mers in hash buckets
  - **collisions**
  - below 100% load (empty buckets)

Part II: How to Hash

Collision Resolution Strategies

- Chaining (shown): use linked lists at each address
- Open addressing with linear probing:
  relocate colliding keys to following addresses
- Open addressing with non-linear (quadratic) probing:
  relocate colliding keys to other addresses, non-linearly
- Double hashing: relocate colliding keys by linear probing
  with different step sizes for each key
- Cuckoo hashing: use two hash functions, move keys around
- Robin Hood hashing: use linear probing,
  move keys around such that each key stays close to its hash address

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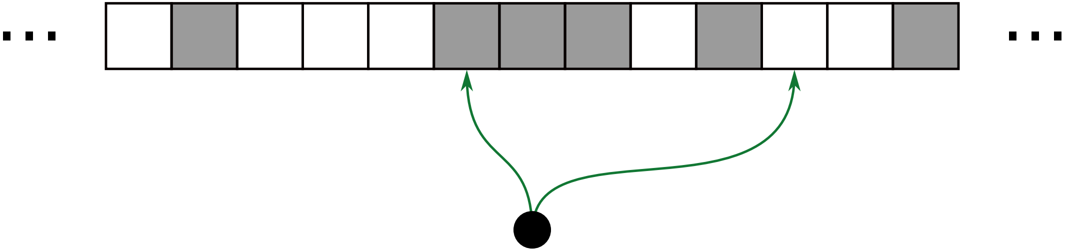
Requirements for Genome Analysis

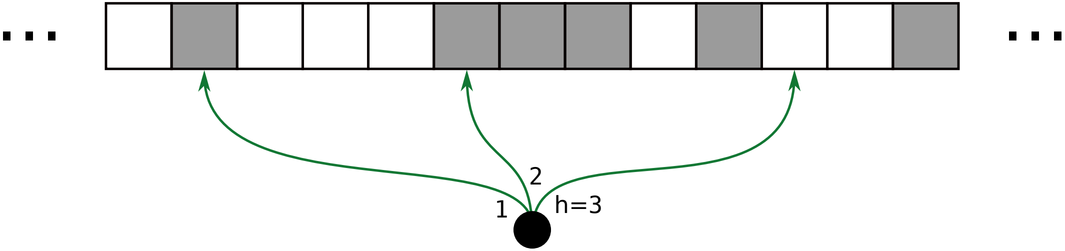
- Very fast lookup, i.e., very few random memory accesses.
- Small size, i.e., high load factor, almost no empty space

(h, b) Cuckoo hashing



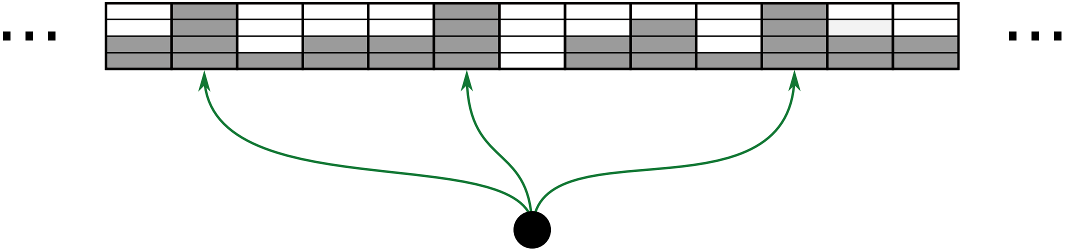
(h, b) Cuckoo hashing

- Use h hash functions



(h, b) Cuckoo hashing

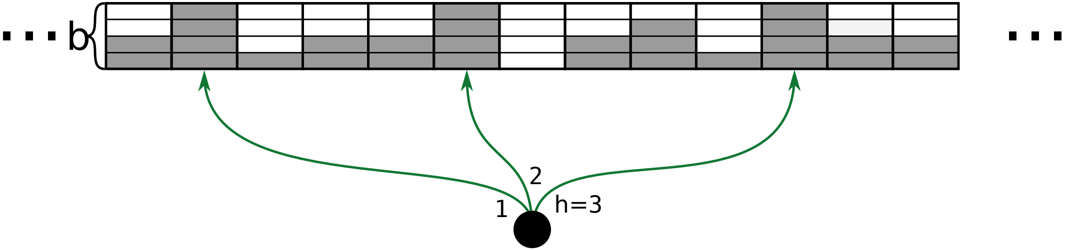
- Use h hash functions
- Store up to b elements at each position



(*h*, *b*) Cuckoo hashing

- Use *h* hash functions
- Store up to *b* elements at each position

`load factor =` $$\frac{\#\blacksquare}{\#\square + \#\blacksquare}$$



(h, b) Cuckoo Hashing: Insertion by Random Walk

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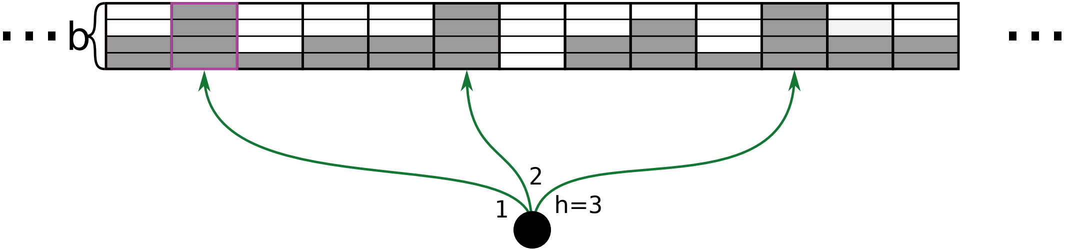

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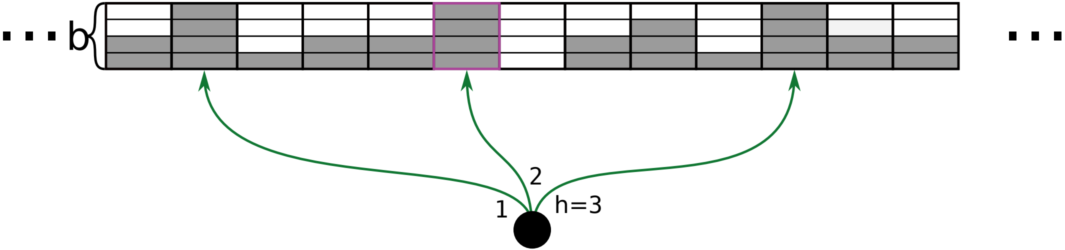

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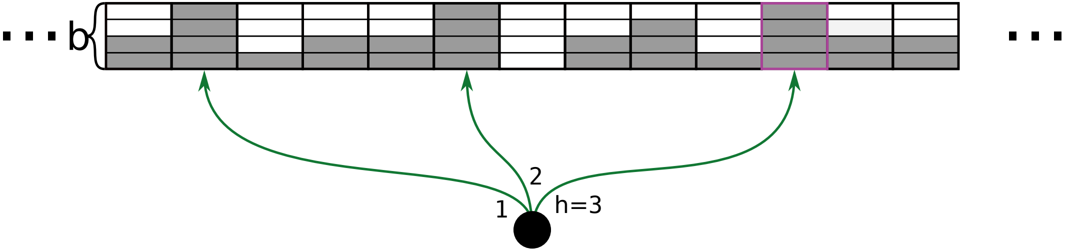

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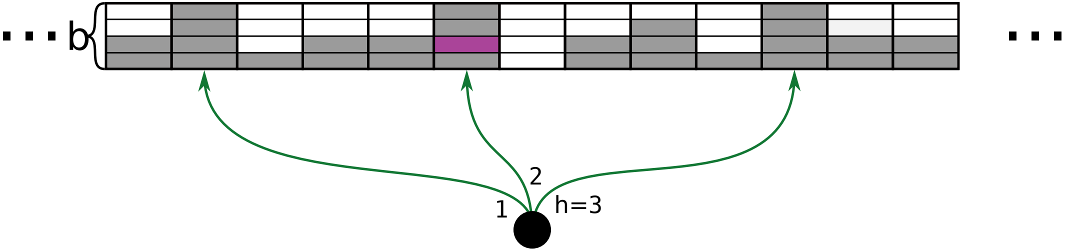

Properties of (h, b) Cuckoo Hashing

- Small number of hash functions: $h = 2, 3, 4$
- Number h limits random memory accesses (cache misses) per lookup
- Higher h allow higher loads with same bucket size b
- Bucket size b limits number of comparisons per bucket (fast anyway)
- Higher bucket size allows higher loads
- Use $(2, 6)$ or $(3, 4)$ in practice
- Low loads: Many keys found at first choice (fast)
- High loads: Frequently have to check 2nd/3rd choice (slower), but less wasted space
- Good load factors in practice: 0.85 to 0.95

Part III: Similarities and Distances

Similarity vs. Distance

Relationship

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Hamming distance

$P_1 = \texttt{ABCDE}$
$P_2 = \texttt{ABDDE}$
$d(P_1, P_2) = 1$
$\hat{d}(P_1, P_2) = \frac{d(P_1,P_2)}{\ell} = \frac{1}{5}$,
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$P_1 = \texttt{ABCDE}$

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$S(P_1, P_2) = 4$

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$$\hat{S}(P_1, P_2) = 1 - \hat{d}(P_1, P_2)$$

Distance Measures: ℓ_p Distances

Definition

In an ***n*-dimensional real vector space**, points are vectors of n real numbers. For any constant $p \geq 1$, we define the **ℓ_p distance** by

$$d_p([x_1, \cdots, x_n], [y_1, \cdots, y_n]) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$$

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

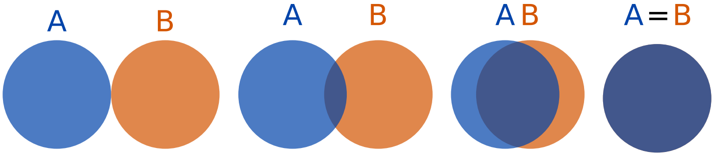
$$d_\infty(x, y) = \max_{i=1,\ldots,n} |x_i - y_i|$$

Distance Measure: Jaccard Similarity

Definition

Given two sets A and B, the **Jaccard index** or **Jaccard similarity** is defined as

$$J(A,B) = \frac{|A \cap B|}{|A \cup B|}.$$



$$0 \leq J(A,B) \leq 1$$

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Jaccard Distance

We define the **Jaccard distance** as

$$d_J(A, B) := 1 - J(A, B) \,,$$

which is a metric.

Reminder: Hamming Distance

Setting

Compare two strings $s, t \in \Sigma^n$ of same length.

Definition

Given $s, t \in \Sigma^n$, we define the **Hamming distance** $d_H(s,t)$ as the number of positions where s and t differ. In other words,

$$d_H(s,t) := \left|\left\{i \in \{0, \ldots, n-1\} : s[i] \neq t[i]\right\}\right|$$

Example

$$\begin{matrix} s = \\ t = \end{matrix}\ \begin{matrix} \texttt{C} \\ \texttt{C} \end{matrix}\ \boxed{\begin{matrix} \texttt{T} \\ \texttt{A} \end{matrix}}\ \begin{matrix} \texttt{G} \\ \texttt{G} \end{matrix}\ \begin{matrix} \texttt{T} \\ \texttt{T} \end{matrix}\ \boxed{\begin{matrix} \texttt{A} \\ \texttt{C} \end{matrix}}\ \begin{matrix} \texttt{A} \\ \texttt{A} \end{matrix}\ \begin{matrix} \texttt{T} \\ \texttt{T} \end{matrix}\ \begin{matrix} \texttt{A} \\ \texttt{A} \end{matrix}\ \begin{matrix} \texttt{C} \\ \texttt{C} \end{matrix}$$

$\Rightarrow$ Hamming distance 2

Reminder: Edit Distance

Edit Operations

- Replacement:

  C T G T A A T A C

  C A G T A A T A C

- Insertion:

  C T G T A A T A C

  C T G T C A A T A C

- Deletion:

  C T G T A A T A C

  C T G T A T A C

Definition: edit distance

The **edit distance** of s and t is defined as the **minimum** number of **edit operations** needed to turn s into t.

Problem: Finding Similar Items

Setting

Suppose $\mathcal{U}$ is a universe (set) of objects and d is a metric on $\mathcal{U}$.

Problem

Given a set $A \subseteq \mathcal{U}$, an item x, and $\epsilon > 0$.
Find all items similar to x in A, i.e., all items $e \in A$ such that $d(e, x) < \epsilon$.

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Naïve approach

```
def find_similar(A, x, eps):
    for element in A:
        if d(element, x) < eps:
            yield element
```

Time complexity: $O(|A| \cdot \dim \mathcal{U})$

Part IV: Locality Sensitive Hashing

Hashing

Observation

Conventional hash functions are designed to generate scattered hash values even for **similar** (not identical) items.

Argument

It is necessary:

- Avoid **collisions** as much as possible,
- Preserve **constant time** lookup operation in exact membership query.

Sought

Find a hashing technique to give two similar items an identical hash value.

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Find a hashing technique to give two similar items an identical hash value.
Obvious transitivity problem!

Locality Sensitive Hashing

Idea

- Design hash functions that tend to assign **identical hash values** for **similar items** *a* and *b* with high probability.
- **Collision:** items **may** be similar.
- Distinct hash values: items may also be similar ?!

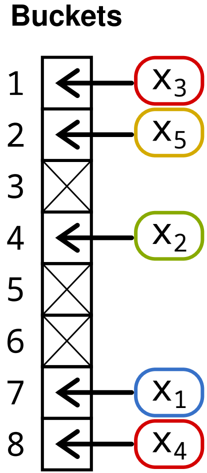
Preprocessing

- Compute hash value for each element in the set.
- Put element in the corresponding bucket.

Querying

- Compute the hash value for query item.
- Compare the query item only with items in its corresponding bucket.

Locality Sensitive Hashing



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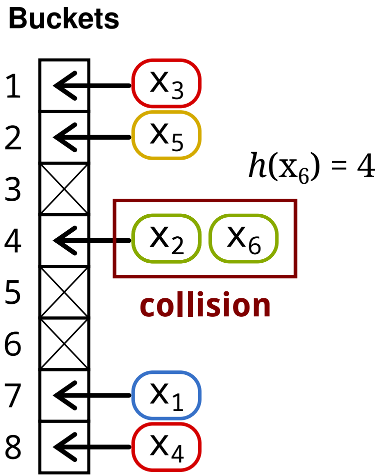

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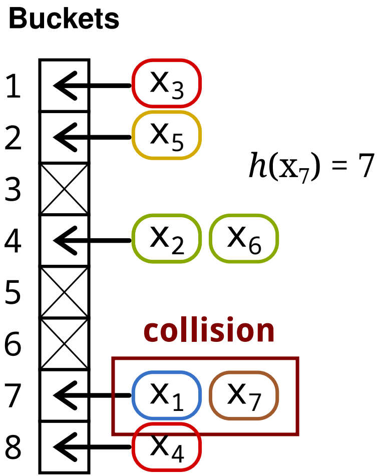


$h(x_7) = 7$

Locality Sensitive Hashing

Definition: Locality Sensitive Hashing

Let $\mathcal{S}$ be a similarity measure on space or universe $\mathcal{U}$.
A set $\mathcal{H}$ of hash functions is **locality sensitive** for $\mathcal{S}$ if

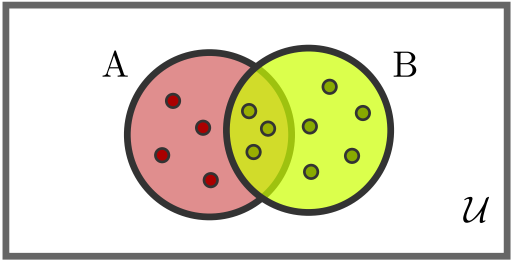
$$\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \mathcal{S}(x, y) \quad \text{for all } x, y\,,$$

where the probability is taken over the distribution of hash functions.

Example: Hamming similarity

Consider the set of hash functions $\mathcal{H} = \{P_i | i \in 1..n\}$, where $P_i(s_1 s_2 \ldots s_n) := s_i$.
Then $\Pr[h(x) = h(y)] = \mathcal{S}_{\text{Hamming}}(x, y)$.
Therefore $\mathcal{H}$ is a LS set of hash functions for $\mathcal{S}_{\text{Hamming}}$.

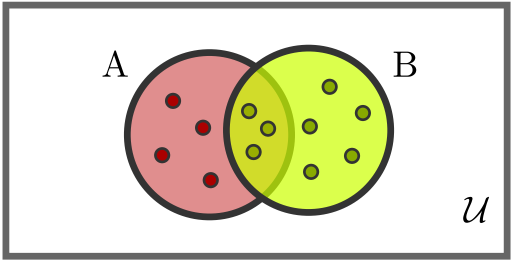
LSH for Jaccard Similarity



$$\mathcal{S}_J = \frac{|A \cap B|}{|A \cup B|}$$

$$\mathcal{S}_J(A, B) = \frac{3}{12} = 0.25$$

LSH for Jaccard Similarity



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Idea

A bijective function $\pi : \mathcal{U} \to [0, |\mathcal{U}|[$ is a ranking (ordering) function of $\mathcal{U}$.
The family $\mathcal{H}$ of hash functions

$$h_\pi(A) := \min_{x \in A} \pi(x)\,,$$

where π ranges over **all orderings** of $\mathcal{U}$, is **locality sensitive** for $\mathcal{S}_J$.

Observations on Locality Sensitive Hashing

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Idea (amplification)

Use **multiple hash functions**,
hoping that all similar elements get identical value for at least one hash function.

Sketches

Definition

For an element x, a **sketch** for the LSH $\mathcal{H}$ is a vector $[h_1(x), h_2(x), \ldots, h_r(x)]$, where hash functions h_i are selected from $\mathcal{H}$ according to a probability distribution.

Two benefits of sketches

1. Increased chance of finding a similar item
   when searching with more hash functions
2. Estimation of similarity: $|\{i \mid h_i(A) = h_i(B)\}|/r$ is an estimate of $S(A, B)$:
   Using only one hash function gives a **high-variance** estimator.
   Using more hash functions gives higher precision.

Error Rates with Sketches (Several Hash Functions)

- False negative errors decrease exponentially with r
- False positive errors increase slowly linearly with r

Summary

Hashing

- Alternative to classical k-mer index for large k
- Requires a collision resolution strategy
- Good in practice: (h, b) Cuckoo hashing: several hash functions, buckets of size b

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Locality Sensitive Hashing

- Different similarity measures
- Probability that hash values of x, y agree $=$ similarity of x, y
- sets of k-mers: Jaccard similarity
- min hashing
- amplification using several hash functions; sketches

Possible exam questions

- How can a k-mer index be implemented?
- What is the disadvantage of hash-based vs. encoding-based implementations?
- How can k-mers be mapped bijectively to the integers $0, \dots, 4^k - 1$?
- What are some common collision resolution strategies when hashing?
- Explain (h, b) Cuckoo hashing
- Why are the advantages and disadvantages of (h, b) Cuckoo hashing?
- When is a set of hash functions "locality sensitive"?
- Why is standard hashing usually not locality sensitive?
- Explain min-hashing.
- Why is min-hashing locality sensitive for the Jaccard similarity?